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Singular perturbation theory applied to the collective oscillation of gas bubbles in a liquid

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By using the technique of multiple scaling a theory of gas-bubble oscillations in a liquid is developed. Collections of bubbles of arbitrary shape under the action of surface tension, buoyancy and solid surfaces are considered. In the absence of thermal conduction in the bubbles and compressibility of the liquid a conserved 'action' is defined for each of the modes of oscillation. The equation governing the decay of the action with time is found by carrying the analysis to second order. The geometrical configuration of the bubbles, in which the oscillations take place, evolves in time under convection by an underlying 'basic' flow for which the governing equations are derived. The bubble pulsations influence the development of the basic motion. Later in the work a source of gas bubbles is brought in and its effects on the oscillations discussed. The results of the interaction of pulsating bubbles with the liquid surface are also briefly considered. The determination of the amplitude of oscillations induced by the splitting up of bubbles and by the generation of bubbles from the gas source is described. Finally, several applications of the theory to specific problems are given.

1. Introduction

The object of this paper is to show how a theory of gas bubbles in a liquid can be developed using singular perturbation theory. We use formally the popular techniques of multiple scaling and averaging to separate the rapid oscillations of the bubbles from the slower 'basic' motion which underlies them.

The oscillations themselves are described by a certain eigenvalue problem, whose solution gives the normal modes and corresponding frequencies of oscillation. The decay of the bubble pulsations under the effect of thermal conduction in the interior of the bubbles and compressibility of the liquid is described by an amplitude equation for each of the modes.

The basic motion of the liquid is modelled by an incompressible flow with certain boundary conditions on the bubble surfaces. In general, it is found that the oscillations affect the basic flow, but since the equations governing the latter are intractable, we place the emphasis on a description of the bubble oscillations for a given basic motion.

If a bubble breaks into two, under the action of the underlying motion, it is found that bubble oscillations are generated. We provide a method for determining the amplitude of the resulting modes of pulsation.

In the following sections we present the theory for a collection of bubbles in a liquid. We allow for the presence of solid surfaces, surface tension and buoyancy effects.

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Later on, we will also introduce a mechanism for production of the bubbles and will briefly examine the effects of introducing an upper surface to the liquid.

Much previous work on bubble oscillations has concentrated on single bubbles, or on the averaged properties of a bubbly medium. Here we refer the reader to the papers by Foldy (1945), Whitfield & Howe (1976) and the review articles of Fitzpatrick & Strasberg (1957), van Wijngaarden (1972) and Plesset & Prosperetti (1977). The present study is concerned with the collective pulsations of finite groups of bubbles, such as might be found near a source of gas in a liquid or after the break up of a large bubble into smaller ones. It should be emphasised that our goal is not to deal with the continuum limit of waves in a bubbly medium, a limit which has already been extensively studied elsewhere.

Finally, a survey of multiple-scaling techniques can be found in Nayfeh (1973).

2. Basic equations

We neglect the viscosity of the gas and liquid, and we assume the former to be polytropic and the latter to be almost incompressible. Damping of the bubble oscillations due to thermal conduction, viscosity and compressibility of the liquid (radiative damping) was studied by Devin (1959), amongst others. He concludes that viscous damping is unimportant unless the bubbles are very small; however, radiative damping becomes significant when the bubble size is above about 1 mm for air bubbles in water at one atmosphere. We will therefore neglect viscosity throughout, although it should be borne in mind that it may affect the (slower) basic motion. At this stage we will take the liquid to be incompressible; later on we will see that slight compressibility can be introduced to account for radiative damping of the bubble pulsations. However, we will always take the group of bubbles to be small compared to the wavelength associated with the bubble oscillations. This is not a serious restriction unless the number of bubbles is very large, because each bubble is small on a wavelength scale.

Each bubble is labelled by a positive integer $j = 1, \ldots, N$ and the interior of the bubble j will be denoted by V^j and its surface by J^j. That part of J^j which is in contact with the liquid is S^j, while to allow for the possibility that the bubble has an interface with a solid boundary we write this part of the bubble surface as Σ^j; the latter can be empty, in the sense of set theory, in which case the bubble does not touch the solid. We will also use the above symbols to denote the volume or surface area of the corresponding regions. Finally, we write V for the region occupied by the liquid and S for the liquid–solid interface.

We take S^j to have the form

$$f^j(\mathbf{x}, t) = 0, \tag{2.1}$$

so that we have the kinematic relations

$$\frac{\partial f^j}{\partial t} + \nabla\phi \,.\, \nabla f^j = 0, \quad (\nabla\phi - \nabla\phi^j) \,.\, \nabla f^j = 0 \quad \text{on } S^j, \qquad (2.2), (2.3)$$

where ϕ and ϕ^j represent the velocity potentials of the liquid and gas respectively, so that

$$\mathbf{u} = \nabla\phi, \quad \mathbf{u}^j = \nabla\phi^j \tag{2.4}$$

are the corresponding velocities. We assume the liquid flow to be steady at infinity, so that the pressure of the liquid takes the Bernoulli form

$$p = p_\infty - \rho_l(\tfrac{1}{2}|\nabla\phi|^2 + \partial\phi/\partial t) - \rho_l gy, \tag{2.5}$$

where ρ_l is the density of the liquid and y a vertical co-ordinate.

We take as non-dimensional variables

$$\mathbf{x}' = \mathbf{x}/a, \quad t' = t\left(\frac{p_\infty}{\rho_l a^2}\right)^{\frac{1}{2}}, \quad \mathbf{u}' = \left(\frac{a\rho_l}{T}\right)^{\frac{1}{2}}\mathbf{u},$$

$$\phi' = \left(\frac{\rho_l}{aT}\right)^{\frac{1}{2}}\phi, \quad p' = (p - p_\infty)\left(\frac{a}{p_\infty T}\right)^{\frac{1}{2}},$$

$$\rho' = \left(\frac{R\theta_\infty}{p_\infty}\rho - 1\right)\left(\frac{ap_\infty}{T}\right)^{\frac{1}{2}},$$

$$\theta' = \left(\frac{\theta}{\theta_\infty} - 1\right)\left(\frac{ap_\infty}{T}\right)^{\frac{1}{2}},$$

where T is the surface tension of the liquid, R the gas constant, θ_∞ the ambient temperature and a is a typical bubble size, not to be confused with the same symbol which is used for the action later in this article. The time scale $(\rho_l a^2/p_\infty)^{\frac{1}{2}}$ is based on a typical period of oscillation of the bubbles while the scaling $(T/a\rho_l)^{\frac{1}{2}}$ for the velocity was chosen to make the pressure variations of the basic flow of order T/a. This condition is necessary, but not always sufficient, to keep the bubbles stable, otherwise they will break up into smaller ones for which the condition *is* satisfied.

The equations governing the motion are the following:

$$\nabla^2\phi = 0, \quad p = -\frac{\partial\phi}{\partial t} - \frac{1}{2}\epsilon|\nabla\phi|^2 - \Delta y, \tag{2.6), (2.7}$$

in the liquid, and

$$\mathbf{u}^j = \nabla\phi^j, \quad \delta(1+\epsilon\rho^j)\left(\frac{\partial\mathbf{u}^j}{\partial t} + \epsilon\mathbf{u}^j.\nabla\mathbf{u}^j + \Delta\mathbf{e}_y\right) = -\nabla p^j \tag{2.8), (2.9}$$

$$p^j = \rho^j + \theta^j + \epsilon\rho^j\theta^j, \tag{2.10}$$

$$\frac{\partial\rho^j}{\partial t} + \epsilon\mathbf{u}^j.\nabla\rho^j + (1+\epsilon\rho^j)\nabla.\mathbf{u}^j = 0, \tag{2.11}$$

$$(1+\epsilon\rho^j)\left(\frac{\partial\theta^j}{\partial t} + \epsilon\mathbf{u}^j.\nabla\theta^j\right) - (\gamma-1)(1+\epsilon\theta^j)\left(\frac{\partial\rho^j}{\partial t} + \epsilon\mathbf{u}^j.\nabla\rho^j\right) = \eta^2\nabla^2\theta^j, \tag{2.12}$$

in the gas. In these equations, and in the remainder of this paper, we have dropped the prime on each variable and have introduced the parameters

$$\epsilon = \left(\frac{T}{p_\infty a}\right)^{\frac{1}{2}}, \quad \delta = \frac{p_\infty}{R\theta_\infty\rho_l}, \tag{2.13}$$

$$\Delta = \rho_l ga\left(\frac{a}{p_\infty T}\right)^{\frac{1}{2}}, \quad \eta = \left[\frac{\lambda\theta_\infty(\gamma-1)}{ap_\infty}\left(\frac{\rho_l}{p_\infty}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{2.14}$$

where λ is the coefficient of heat conduction of the gas.

The kinematic conditions (2.2) and (2.3) are now

$$\frac{\partial f^j}{\partial t} + \epsilon\nabla\phi.\nabla f^j = 0, \quad (\nabla\phi - \nabla\phi^j).\nabla f^j = 0 \quad \text{on} \quad S^j. \tag{2.15), (2.16}$$

Moreover, on the solid boundary we have

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S, \tag{2.17}$$

$$\frac{\partial \phi^j}{\partial n} = 0 \quad \text{on} \quad \Sigma^j. \tag{2.18}$$

There is a pressure-jump relation across S^j of the form

$$p^j - p = \epsilon\left(\frac{1}{R^j} + \frac{1}{R'^j}\right), \tag{2.19}$$

where R^j and R'^j are the principal radii of curvature of S^j at the point in question.

In principle, we should consider heat-transfer effects in the liquid and solid; these would provide boundary conditions on J^j for the temperature θ^j. However, because of the large heat capacity of both the solid and liquid compared with the gas, we can take the simpler condition that

$$\theta^j = 0 \quad \text{on} \quad J^j. \tag{2.20}$$

A useful relation can be derived using equations (2.10)–(2.12):

$$\frac{d}{dt}\int_{V^j} p^j dv - \int_{S^j} (\gamma + \epsilon(\gamma-1)p^j)\frac{\partial \phi^j}{\partial n} dS - \epsilon(\gamma - 1)\int_{V^j} \mathbf{u}^j . \nabla p^j \, dv = -\eta^2 \int_{J^j} \frac{\partial \theta^j}{\partial n} dS, \tag{2.21}$$

where the normal derivative $\partial/\partial n$ is here, as elsewhere, taken *inwards*, towards the gas.

We have neglected compressibility of the liquid in setting up these equations, this requires that the scaled wavenumber

$$K = \frac{1}{c_0}\left(\frac{p_\infty}{\rho_l}\right), \tag{2.22}$$

where c_0 is the sound speed in the liquid, be very small. Later in the article we will include the effects of liquid compressibility on the bubble oscillations.

In §3 we proceed to apply multiple-scaling ideas to this problem.

3. The perturbation procedure

We assume that the parameters ϵ, δ, Δ and η are all small and, to simplify the analysis, are formally of the same order. The parameters have the following meanings.

(*a*) ϵ is a measure of bubble stiffness, that is the pressure required to change the bubble's shape relative to that required to change its volume; if it were not small then we would not be able to separate volume pulsations from the general oscillation of the bubble surface.

(*b*) δ is the ratio of gas density to liquid density and so is generally small.

(*c*) Δ/ϵ is a measure of the effectiveness of buoyancy compared to surface tension forces – the bubbles would be structurally unstable if it were large and so Δ will be small provided ϵ is.

(*d*) η measures the effects of damping of the oscillations due to thermal conduction in the gas – if it is $O(1)$ then any volume pulsations are damped out very quickly, while if it is large then the gas motion is almost isothermal (a problem which can be treated by the techniques of this paper), and damping due to thermal conduction is again small. Taking air bubbles in water at one atmosphere, all four parameters are small for bubble radii between about ten microns and one centimetre – furthermore, each of the parameters, apart from δ, gets even smaller at higher ambient pressures and δ remains small up to pressures of a few hundred atmospheres.

The expansions all take the form

$$\psi = \psi_0 + \epsilon\psi_1 + o(\epsilon), \tag{3.1}$$

and we introduce the two time scales

$$t_0 = t, \quad t_1 = \epsilon t, \tag{3.2}$$

the former being the fast time scale, representing the oscillations, and the latter being the slow time scale, representing the basic flow. At order ϵ^l we will write $\overline{\psi}_l$ for the average of ψ_l with respect to t_0 and $\tilde{\psi}_l = \psi_l - \overline{\psi}_l$ for the corresponding oscillation in ψ_l. Averaging, in this way, is a convenient way of eliminating secular terms in the expansion and, at the same time, separating the bubble pulsations from the basic flow.

Substituting expansions of the form (3.1) into the governing equations, we find that

$$\nabla^2\phi_0 = 0, \quad p_0 = -\frac{\partial\phi_0}{\partial t_0}, \tag{3.3), (3.4}$$

$$\nabla p_0^j = 0, \quad p_0^j = \rho_0^j + \theta_0^j, \tag{3.5), (3.6}$$

$$\frac{\partial\theta_0^j}{\partial t_0} = (\gamma - 1)\frac{\partial\rho_0^j}{\partial t_0}, \tag{3.7}$$

$$\nabla . \mathbf{u}_0^j = -\frac{\partial\rho_0^j}{\partial t_0}, \tag{3.8}$$

while (2.15)–(2.19) yield

$$\frac{\partial f_0^j}{\partial t_0} = 0, \tag{3.9}$$

$$\left.\begin{aligned}(\nabla\phi_0 - \nabla\phi_0^j) . \nabla f_0^j &= 0\\ p_0^j &= p_0\end{aligned}\right\} \quad \text{on} \quad S_0^j, \tag{3.10}$$

$$\frac{\partial\phi_0}{\partial n} = 0 \quad \text{on} \quad S_0, \tag{3.11}$$

$$\frac{\partial\phi_0^j}{\partial n} = 0 \quad \text{on} \quad \Sigma_0^j. \tag{3.12}$$

Here S_0, S_0^j and Σ_0^j are the zeroth-order approximations to S, S^j and Σ^j respectively; these are independent of t_0 by equation (3.9), which is equivalent to the fact that, in linear theory, boundary conditions should be imposed on the unperturbed surface.

Using (3.5)–(3.7) we deduce that

$$\tilde{p}_0^j = \gamma\tilde{\rho}_0^j = \tilde{p}_0^j(t), \tag{3.13}$$

so that, by (3.8), we have

$$\gamma\nabla . \mathbf{u}_0^j = -\frac{d\tilde{p}_0^j}{dt_0}, \tag{3.14}$$

which, integrated over V_0^j, yields the equation

$$\gamma\int_{J_0^j}\frac{\partial\tilde{\phi}_0^j}{\partial n}dS = V_0^j\frac{d\tilde{p}_0^j}{dt_0}. \tag{3.15}$$

Using equations (3.4), (3.10) and (3.12) this relation becomes

$$\gamma \int_{S_0^j} \frac{\partial \tilde{\phi}_0}{\partial n} dS + V_0^j \frac{\partial^2 \tilde{\phi}_0}{\partial t_0^2} = 0 \quad \text{on} \quad S_0^j. \tag{3.16}$$

From equations (3.3) and (3.11) we obtain

$$\nabla^2 \tilde{\phi}_0 = 0, \tag{3.17}$$

$$\frac{\partial \tilde{\phi}_0}{\partial n} = 0 \quad \text{on} \quad S_0. \tag{3.18}$$

These, together with equation (3.16), govern the time development of $\tilde{\phi}_0$. As expected, at zero order we have just the linearized problem.

As is usual in such problems (Nayfeh & Mook 1979), we look for normal modes of the form

$$\tilde{\phi}_0 = \Phi(\mathbf{x}) \exp(-i\Gamma(t_1)/\epsilon), \tag{3.19}$$

and then the problem becomes

$$\nabla^2 \Phi = 0, \tag{3.20}$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on} \quad S_0, \tag{3.21}$$

$$\frac{\gamma}{V_0^j} \int_{S_0^j} \frac{\partial \Phi}{\partial n} dS = \omega^2 \Phi \quad \text{on} \quad S_0^j. \tag{3.22}$$

Here we have written

$$\omega = \frac{d\Gamma_1}{dt_1} \tag{3.23}$$

for the 'frequency' of the mode in question. Let Ω^j be the solution of (3.20) and (3.21) in V_0 for which

$$\Omega^j = \delta_{jk} \quad \text{on} \quad S_0^k. \tag{3.24}$$

We can write Φ in the form

$$\Phi = \sum_j \alpha^j \Omega^j, \tag{3.25}$$

and then the problem for Φ takes the form of the system of linear equations

$$\frac{\gamma}{V_0^j} \sum_{k=1}^{N} \int_{S_0^j} \frac{\partial \Omega^k}{\partial n} dS\, \alpha^k = \omega^2 \alpha^j \tag{3.26}$$

for the coefficients α^j. Set $\beta^j = (V_0^j)^{\frac{1}{2}} \alpha^j$ and define the matrix

$$A_{jk} = \frac{\gamma}{(V_0^j V_0^k)^{\frac{1}{2}}} \int_{S_0^j} \frac{\partial \Omega^k}{\partial n} dS. \tag{3.27}$$

Then we have

$$\sum_{k=1}^{N} A_{jk} \beta^k = \omega^2 \beta^j, \tag{3.28}$$

which is now in the standard eigenvalue form, the eigenvalue being ω^2 and the eigenvector being β^j. Using Green's formula, equation (3.27) becomes

$$A_{jk} = \frac{\gamma}{(V_0^j V_0^k)^{\frac{1}{2}}} \int_{V_0} \nabla \Omega_j . \nabla \Omega_k \, dV, \tag{3.29}$$

which is obviously symmetric and positive-definite. There are therefore N real eigenfrequencies $\omega_\mu(\mu = 1, \ldots, N)$, which are chosen positive, with corresponding orthonormal eigenvectors β^j_μ, i.e.

$$\sum_{j=1}^{N} \beta^j_\mu \beta^j_\nu = \delta_{\mu\nu}.$$

We now have the general solution for $\tilde{\phi}_0$ in the form

$$\tilde{\phi}_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} A_\mu(t_1) \exp\left(-i\Gamma_\mu(t_1)/\epsilon\right) \Phi_\mu(\mathbf{x})\right\}, \tag{3.30}$$

where A_μ is the amplitude of the mode labelled by μ and

$$\frac{d\Gamma_\mu}{dt_1} = \omega_\mu(t_1). \tag{3.31}$$

The eigenvalue problem (3.28) changes on the slow time scale t_1 as the geometry of the bubbles, represented by S^j_0, changes under the influence of the basic flow. The modes represent collective oscillations of the system of bubbles and cannot, in general, be assigned to particular bubbles in any natural way.

We note here that the determination of the matrix A_{jk} for a given bubble geometry is closely related to a classical problem of electrostatic theory. If there are no solid boundaries present and we replace the bubbles by conductors *in vacuo* then the matrix $\mathbf{A}$ is simply related to the capacitance matrix for this system of conductors. Solid surfaces have no simple analogue, but the techniques of electrostatics can be usefully carried over to the study of bubble oscillations. We will give examples of this later in this article.

Returning to the formal theory, equations (3.4) and (3.30) give

$$\tilde{p}_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} i\omega_\mu A_\mu \exp\left(-i\Gamma_\mu/\epsilon\right) \Phi_\mu\right\}, \tag{3.32}$$

while equations (3.7), (3.10) and (3.13) yield

$$\tilde{p}^j_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} i\omega_\mu A_\mu \exp\left(-i\Gamma_\mu/\epsilon\right) \alpha^j_\mu\right\}, \tag{3.33}$$

$$\tilde{\rho}^j_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} \frac{i\omega_\mu}{\gamma} A_\mu \exp\left(-i\Gamma_\mu/\epsilon\right) \alpha^j_\mu\right\}, \tag{3.34}$$

$$\tilde{\theta}^j_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} \frac{i(\gamma-1)\,\omega_\mu}{\gamma} A_\mu \exp\left(-i\Gamma_\mu/\epsilon\right) \alpha^j_\mu\right\}. \tag{3.35}$$

We next define $\Phi^j_\mu(\mathbf{x})$ in V^j_0 as follows: Φ^j_μ satisfies

$$\nabla^2 \Phi^j_\mu = -\frac{\omega^2_\mu \alpha^j_\mu}{\gamma}, \tag{3.36}$$

$$\frac{\partial \Phi^j_\mu}{\partial n} = \frac{\partial \Phi_\mu}{\partial n} \quad \text{on} \quad S^j_0, \tag{3.37}$$

$$\frac{\partial \Phi^j_\mu}{\partial n} = 0 \quad \text{on} \quad \Sigma^j_0, \quad \int_{V^j_0} \Phi^j_\mu \, dv = 0, \tag{3.38}$$

so that the solution of (3.14) for $\tilde{\phi}_0^j$ is given by

$$\tilde{\phi}_0^j = \mathscr{R}\left\{\sum_{\mu=1}^{N} A_\mu \exp(-i\Gamma_\mu/\epsilon)\,\Phi_\mu^j\right\}. \tag{3.39}$$

We now have expressions for all the oscillating quantities $\tilde{\phi}_0$, $\tilde{p}_0$, $\tilde{p}_0^j$, $\tilde{\rho}_0^j$, $\tilde{\theta}_0^j$ and $\tilde{\phi}_0^j$ in the forms (3.30), (3.32)–(3.35) and (3.39). As regards the averaged quantities, we deduce that

$$\nabla^2\overline{\phi}_0 = 0, \quad \overline{p}_0 = 0 \qquad (3.40), (3.41)$$

from equations (3.3) and (3.4), and that

$$\overline{p}_0^j = \overline{\rho}_0^j + \overline{\theta}_0^j = 0, \quad \nabla.\overline{\mathbf{u}}_0^j = \nabla^2\overline{\phi}_0^j = 0 \qquad (3.42), (3.43)$$

from equations (3.5), (3.6), (3.8), (3.10) and (3.41). The equations (3.15) and (3.43) yield the relations

$$\frac{dV_0^j}{dt_1} = 0, \quad \tilde{V}_1^j = -\frac{V_0^j}{\gamma}\tilde{p}_0^j, \qquad (3.44), (3.45)$$

where V_1^j is the second term in the expansion

$$V^j = V_0^j + \epsilon V_1^j + o(\epsilon) \tag{3.46}$$

of the volume of bubble j.

We now draw attention to the presence of a singularity of the expansions in the form of a thermal boundary layer in the bubble close to its surface. This occurs because $\tilde{\theta}_0^j$, as given by equation (3.35), does not in general satisfy the condition (2.20). However, the situation is easily remedied by the introduction of a rescaling based on the thermal boundary-layer thickness η. We therefore let the distance normal to the bubble surface be $\eta\xi$, so that ξ is a co-ordinate appropriate to the boundary layer. Performing expansions of the form

$$\psi^j = \psi_0^{j*} + o(1), \tag{3.47}$$

for each of the quantities p^j, ρ^j, θ^j and $\mathbf{u}^j$ in the boundary layer and matching to the region inside the bubble, we find that

$$p_0^{j*} = p_0^j(t) = \theta_0^{j*} + \rho_0^{j*}, \quad \mathbf{u}_0^{j*} = \mathbf{u}_0^j, \qquad (3.48), (3.49)$$

$$\gamma\frac{\partial\theta_0^{j*}}{\partial t_0} - \frac{\partial^2\theta_0^{j*}}{\partial\xi^2} = (\gamma-1)\frac{dp_0^j}{dt_0}. \tag{3.50}$$

The solution of this problem is easy; we find that

$$\tilde{\theta}_0^{j*} = \mathscr{R}\left\{\sum_{\mu=1}^{N}\frac{i(\gamma-1)\,\omega_\mu}{\gamma}A_\mu\exp(-i\Gamma_\mu/\epsilon)\,\alpha_\mu^j(1-\exp(-\sigma_\mu\xi))\right\}, \tag{3.51}$$

where $\sigma_\mu = (\frac{1}{2}\omega_\mu\gamma)^{\frac{1}{2}}(1-i)$ and we have used the condition (2.20). The other thermodynamic quantities are now given by

$$\tilde{p}_0^{j*} = \tilde{p}_0^j = \tilde{\theta}_0^{j*} + \tilde{\rho}_0^{j*}, \tag{3.52}$$

$$\overline{p}_0^{j*} = \overline{\theta}_0^{j*} = \overline{\rho}_0^{j*} = 0. \tag{3.53}$$

In the §4 we determine the amplitude equation governing the development of the oscillations on the slow time scale t_1.

4. The amplitude equation

As is shown in appendix A, by going to the next order we obtain the governing problem for $\tilde{\phi}_1$, the first-order oscillating velocity potential in the liquid, having the form of equations (3.16)–(3.18), but with a forcing term on the right-hand side of equation (3.16). This forcing term contains sum and difference frequencies corresponding to nonlinear effects and, assuming that (i) no two modes have the same frequency, i.e. we exclude the degenerate case; (ii) the sum and difference frequencies do not coincide with an eigenfrequency, i.e. we do not allow resonant nonlinear coupling of the oscillations; then the solubility of this problem requires that, as usual in multiple-scaling studies, we satisfy an orthogonality condition. This condition yields the amplitude equation (A 25), which describes the development of A_μ with the slow time t_1.

Writing

$$A_\mu = \omega_\mu^{-\frac{1}{2}} a_\mu \exp(-i\tau_\mu), \tag{4.1}$$

we find that a_μ, which we will refer to as the *action*, satisfies

$$\frac{da_\mu}{dt_1} + a_\mu \frac{\omega_\mu^{\frac{1}{2}} \eta(\gamma-1)}{2^{\frac{3}{2}}\gamma^{\frac{1}{2}}\epsilon}\left(\sum_{j=1}^{N} (\alpha_\mu^j)^2 J_0^j\right) = 0, \tag{4.2}$$

and so decays under the effects of thermal conduction in the bubbles. In the absence of thermal conduction, the action of each mode is conserved; this ties in with the general theories of adiabatic invariants for oscillators (see Whitham 1974).

The equation satisfied by τ_μ is more complicated and, since variation of $\tau_{\mu z}$ corresponds to a small correction to the frequency of oscillation, is of little interest.

Up until now, the only source of damping for the oscillations has been thermal conduction in the bubbles. At this point we introduce compressibility of the liquid so that radiative damping also occurs. To this end we take K, the parameter defined by equation (2.22), to be $O(\epsilon)$. This does not change the equations of motion to the order to which we have been working, instead, it modifies the boundary conditions on $\tilde{\phi}_1$ and $\tilde{p}_1$ at infinity and hence the amplitude equation is different.

In order to describe the radiation field in the liquid we introduce the rescaling

$$\hat{\mathbf{x}} = K\mathbf{x}, \tag{4.3}$$

based on the wavelength, and the expansion

$$\tilde{\phi} = K\hat{\phi}_0(\hat{\mathbf{x}}, t) + o(K) \tag{4.4}$$

for the oscillating velocity potential. Then $\hat{\phi}_0$ will satisfy the wave equation

$$\frac{\partial^2 \hat{\phi}_0}{\partial t_0^2} = \hat{\nabla}^2 \hat{\phi}_0. \tag{4.5}$$

Writing

$$\hat{\phi}_0 = \mathscr{R}\left\{\sum_{\mu=1}^{N} \hat{\Phi}_\mu(\hat{\mathbf{x}}) \exp(-i\Gamma_\mu/\epsilon)\right\}, \tag{4.6}$$

we find that $\hat{\phi}_\mu$ satisfies the Helmholtz equation

$$\hat{\nabla}^2 \hat{\Phi}_\mu + \omega_\mu^2 \hat{\Phi}_\mu = 0, \tag{4.7}$$

together with the matching condition to the bubble region ($\mathbf{x} = O(1)$) and a radiation condition as $|\hat{\mathbf{x}}| \to \infty$.

In the general case, when solid surfaces of size comparable to the wavelength are present, we would have to solve the general diffraction problem, and this effectively precludes us from giving general expressions for the radiative damping. In § 7 we will give examples of problems where this can be done, but for the moment we assume that all solid surfaces are small compared to the wavelength (i.e. compact). In this case we know that

$$\tilde{\phi}_0 \sim \frac{q(t)}{r} \quad \text{as} \quad r = |\mathbf{x}| \to \infty, \tag{4.8}$$

where $q(t)$ can be found from

$$4\pi q = \lim_{r \to \infty} \int_{|\mathbf{x}|=r} \frac{\partial \tilde{\phi}_0}{\partial n} dS = \sum_{j=1}^{N} \int_{S_0^j} \frac{\partial \tilde{\phi}_0}{\partial n} dS, \tag{4.9}$$

which becomes

$$4\pi q = \mathscr{R}\left\{\sum_{\mu=1}^{N} \sum_{j=1}^{N} \frac{A_\mu \omega_\mu^2 V_0^j \alpha_\mu^j}{\gamma} \exp\left(-i\Gamma_\mu/\epsilon\right)\right\}, \tag{4.10}$$

by (3.22) and (3.30). We have, therefore,

$$\tilde{\phi}_0 \sim \mathscr{R}\left\{\sum_{\mu=1}^{N} \sum_{j=1}^{N} \frac{A_\mu \omega_\mu^2 V_0^j \alpha_\mu^j}{4\pi\gamma r} \exp\left(-i\Gamma_\mu/\epsilon\right)\right\}. \tag{4.11}$$

The solution of (4.7) is thus given by

$$\hat{\Phi}_\mu = \frac{A_\mu \omega_\mu^2}{4\pi\gamma \hat{r}} \sum_{j=1}^{N} V_0^j \alpha_\mu^j \exp(i\omega_\mu \hat{r}), \tag{4.12}$$

which represents a simple outgoing wave.

We now use the Van Dyke (1975) matching principle for $\tilde{\phi}$ to $O(\epsilon)$ in both the inner and outer regions to show that

$$\hat{\Phi}_1 \sim \frac{K}{\epsilon} \mathscr{R}\left\{\sum_{\mu=1}^{N} \sum_{j=1}^{N} \frac{iA_\mu \omega_\mu^3 V_0^j \alpha_\mu^j}{4\pi\gamma} \exp\left(-i\Gamma_\mu/\epsilon\right)\right\} \quad \text{as} \quad r \to \infty. \tag{4.13}$$

Thus $\Phi_\mu^{(1)}$ (as defined in equation (A 20)) satisfies the same problem as before, except that now

$$\Phi_\mu^{(1)} \to \frac{iKA_\mu \omega_\mu^3}{4\epsilon\pi\gamma} \sum_{j=1}^{N} V_0^j \alpha_\mu^j \quad \text{as} \quad r \to \infty. \tag{4.14}$$

The orthogonality condition then gives the equation for the action,

$$\frac{da_\mu}{dt_1} + \left[\frac{\omega_\mu^{\frac{1}{2}} \eta(\gamma - 1)}{2^{\frac{3}{2}} \gamma^{\frac{1}{2}} \epsilon} \sum_{j=1}^{N} (\alpha_\mu^j)^2 J_0^j + \frac{K\omega_\mu^4}{8\epsilon\pi\gamma}\left(\sum_{j=1}^{N} V_0^j \alpha_\mu^j\right)^2\right] a_\mu = 0, \tag{4.15}$$

where the additional damping term represents radiative effects.

As we mentioned above it is not possible to give a general equation for the action when the solid surfaces are not compact. In particular examples it is not difficult to derive such an equation provided the outer (diffraction) problem can be solved; we will give two such examples in § 7, but, for the above reasons, we will ignore radiative damping until then.

We return to the question of degeneracy and nonlinear coupling of the modes, that is when

$$\omega_\mu = \omega_\nu \quad (\mu \neq \nu), \quad \omega_\mu + \omega_\nu = \omega_\lambda, \tag{4.16), (4.17}$$

respectively. A detailed analysis, which we shall not give here, shows that coupling of the modes may occur if either (4.16) or (4.17) is satisfied to $O(\epsilon^{\frac{1}{2}})$. In general, this situation will only last for a slow-time period of length $O(\epsilon^{\frac{1}{2}})$, and this is not long enough for nonlinear coupling to produce more than a slight transfer of action among the resonant triad (μ, ν and λ). However, if near-degeneracy occurs, it *is* possible to transfer action amongst the degenerate modes – a significant transfer (i.e. $O(1)$) may occur if (4.16) is satisfied to $O(\epsilon^{\frac{1}{2}})$. A recent study of this type of near resonant interaction between modes has been given by Grimshaw & Allen (1979).

5. The basic flow and bubble break-up

In this section we proceed to find the equations and boundary conditions satisfied by the basic motion underlying the bubble oscillations and then examine the effects of the splitting of one bubble into two.

Averaging (A 13) and (A 15) yields

$$\bar{p}_1 = -\frac{1}{2}\left(|\nabla\bar{\phi}_0|^2 + \overline{\left(\frac{\partial\tilde{\phi}_0}{\partial n}\right)^2}\right) - \frac{\partial\bar{\phi}_0}{\partial t_1} - \frac{\Delta y}{\epsilon} \tag{5.1}$$

and, on S_0^j,

$$\bar{p}_1^j = \bar{p}_1 - \overline{\frac{\tilde{f}_1^j \nabla f_0^j \cdot \nabla\tilde{p}_0}{|\nabla f_0^j|^2}} + \left(\frac{1}{R_0^j} + \frac{1}{R_0'^j}\right),$$

$$= \bar{p}_1 + \overline{\left(\frac{\partial\tilde{\phi}_0}{\partial n}\right)^2} + \left(\frac{1}{R_0^j} + \frac{1}{R_0'^j}\right), \tag{5.2}$$

where we have used (3.4) and (A 18) in the last step. We combine (A 3), (5.1) and (5.2) as

$$\frac{\partial\bar{\phi}_0}{\partial t_1} + \tfrac{1}{2}|\nabla\bar{\phi}_0|^2 - \tfrac{1}{2}\overline{\left(\frac{\partial\tilde{\phi}_0}{\partial n}\right)^2} = \frac{1}{R_0^j} + \frac{1}{R_0'^j} - \frac{\Delta y}{\epsilon} - \bar{F}^j(t_1) \quad \text{on} \quad S_0^j. \tag{5.3}$$

We are now in a position to define the basic flow precisely.

The basic motion is defined by the velocity potential $\bar{\phi}_0$ in V_0 and is incompressible, that is

$$\nabla^2\bar{\phi}_0 = 0. \tag{5.4}$$

Equation (A 17) shows that S_0^j is convected with the basic flow and equation (3.44) says that the volume of S_0^j remains constant during the motion. These results, together with the boundary conditions (5.3) on S_0^j and

$$\frac{\partial\bar{\phi}_0}{\partial n} = 0 \quad \text{on} \quad S_0, \tag{5.5}$$

are sufficient to determine the basic motion and hence the geometry in which the bubble oscillations take place.

The presence in (5.3) of the term involving $\tilde{\phi}_0$ shows that the oscillations can affect the basic flow and this complicates the problem considerably. However, even without these terms (i.e. when the oscillations are small) the problem is intractable analytically and would have to be solved numerically. Note that the term in (5.3) containing $\tilde{\phi}_0$ is equivalent to a body force proportional to $\nabla(\overline{|\nabla\tilde{\phi}_0|^2})$ which the oscillations may be thought of as exerting on the basic motion. This is equivalent to the so-called Bjerknes force (see, for example, Crum 1975).

We pass on now to a discussion of the production of bubble oscillations by break-up of a bubble. If a bubble necks and then splits into two, the eigenfrequencies and modes of oscillation undergo a discontinuous change. Furthermore, although the total velocity potential ϕ_0 must be continuous at break up, there may be a transfer between $\bar{\phi}_0$ and $\tilde{\phi}_0$ which will stimulate oscillation of the bubbles.

To determine the amplitude of oscillation after the break-up of the bubble we use the facts that

$$\int_{S_0^j} \frac{\partial \bar{\phi}_0}{\partial n} dS = 0, \tag{5.6}$$

$$\int_{S_0^j} \frac{\partial \tilde{\phi}_0}{\partial n} dS = \frac{V_0^j}{\gamma} \mathscr{R} \left\{ \sum_{\mu=1}^{N} \omega_\mu^2 A_\mu \alpha_\mu^j \exp(-i\Gamma_\mu/\epsilon) \right\}, \tag{5.7}$$

so that

$$\mathscr{R} \left\{ \sum_{\mu=1}^{N} \omega_\mu^2 A_\mu \alpha_\mu^j \exp(-i\Gamma_\mu/\epsilon) \right\} = \frac{\gamma}{V_0^j} \int_{S_0^j} \frac{\partial \phi_0}{\partial n} dS, \tag{5.8}$$

where all the symbols refer to times just after the bubble break-up. We also have

$$\mathscr{R} \left\{ \sum_{\mu=1}^{N} i\omega_\mu A_\mu \alpha_\mu^j \exp(-i\Gamma_\mu/\epsilon) \right\} = p_0 \quad \text{on} \quad S_0^j \tag{5.9}$$

and, since the right-hand sides of (5.8) and (5.9) are continuous at break-up, they are both known from the solution beforehand.

The equations (5.8) and (5.9) are sufficient to determine A_μ and hence continue the solution for $\tilde{\phi}_0$. The basic flow $\bar{\phi}_0$ is now obtained by writing

$$\bar{\phi}_0 = \phi_0 - \tilde{\phi}_0, \tag{5.10}$$

and so we can continue the solution of the basic motion as well.

It should be noted that bubble oscillations are not always produced by bubble break-up. Take the case of a single bubble in a liquid at rest at infinity, which is not oscillating before break-up and for which the division into two is symmetric about some plane through the bubble. In this case the right-hand sides of (5.8) and (5.9) are zero and so *no* oscillations are stimulated to zeroth order, although of course they may occur at higher orders. If the break-up is asymmetric then oscillations will occur in general.

6. The blowing of the bubbles

In this section we introduce a mechanism for supplying gas to the system. There are obviously many ways of modelling this theoretically; we choose the simple one of gas supply through part of the solid boundary.

Formally, we introduce a volume of gas for which $j = 0$ is the bubble index. There is a mass flux of gas $m(\mathbf{x})$ per unit area on Σ^0. This mass injection is assumed independent of time, but it will depend on which part of Σ^0 is considered. We non-dimensionalize m as

$$m' = \frac{R\theta_\infty}{p_\infty} \left(\frac{a\rho_l}{T} \right)^{\frac{1}{2}} m \tag{6.1}$$

and then as before, we drop the prime in what follows. Writing

$$M = \int_{\Sigma^0} m\,dS, \tag{6.2}$$

for the total mass influx, all the equations of §2 remain valid (if we include $j = 0$ as one of the bubbles), apart from (2.18) which becomes

$$(1+\epsilon\rho^0)\frac{\partial\phi^0}{\partial n} = m \quad \text{on} \quad \Sigma_0. \tag{6.3}$$

Equation (2.21) must also be modified when $j = 0$, when it takes the form

$$\frac{d}{dt}\int_{V^0} p^0\,dv - \int_{S^0}(\gamma+\epsilon(\gamma-1)\,p^0)\frac{\partial\phi^0}{\partial n}dS - \gamma\int_{\Sigma^0}(1+\epsilon p^0)\frac{\partial\phi^0}{\partial n}dS - \epsilon(\gamma-1)\int_{V^0}\mathbf{u}^0.\nabla p^0\,dv = -\eta^2\int_{J^0}\frac{\partial\theta^0}{\partial n}dS, \tag{6.4}$$

in which the third term corresponds to the gas injection.

Turning now to §3, equation (3.12) is changed to

$$\frac{\partial\phi_0^0}{\partial n} = m \quad \text{on} \quad \Sigma_0^0, \tag{6.5}$$

and equation (3.44) becomes

$$\frac{dV_0^0}{dt_1} = M. \tag{6.6}$$

The thermal boundary layer on Σ^0 is changed considerably – in place of (3.50) we have

$$\gamma\left(\frac{\partial\theta_0^{j*}}{\partial t_0} + \frac{\epsilon m}{\eta}\frac{\partial\theta_0^{j*}}{\partial\xi}\right) - \frac{\partial^2\theta_0^{j*}}{\partial\xi^2} = (\gamma-1)\frac{dp_0^j}{dt_0}, \tag{6.7}$$

which has the solution (3.51) provided we write

$$\sigma_\mu = -\frac{\gamma\epsilon m}{2\eta} + \left(\frac{\gamma^2\epsilon^2 m^2}{4\eta^2} - i\omega_\mu\gamma\right)^{\frac{1}{2}}, \tag{6.8}$$

so that σ_μ becomes a function of position on Σ_0^0.

It will be noted that the eigenvalue problem for the bubble oscillations remains unchanged by the presence of the mass injection. There are now $N+1$ pulsation modes, one for each of the N bubbles and one for the gas-supply cavity. We will index the modes by $\mu = 0, \ldots, N$.

The procedure adopted in §4 can be followed as before, with the result that the action satisfies

$$\frac{da_\mu}{dt_1} + a_\mu\frac{\omega_\mu^{\frac{1}{2}}\eta(\gamma-1)}{2^{\frac{3}{2}}\gamma^{\frac{1}{2}}\epsilon}\left(\sum_{j=1}^{N}(\alpha_\mu^j)^2 J_0^j\right) + a_\mu\frac{\eta(\gamma-1)}{2\gamma\epsilon}(\alpha_\mu^0)^2\int_{J_0^0}\mathscr{R}\{\sigma_\mu\}\,dS = 0, \tag{6.9}$$

where the additional term is due to the extra dissipation resulting from the presence of the air-supply cavity V^0.

The basic flow can be described as in §5 with the exception that when $j = 0$ condition (3.44) must be replaced by (6.6) owing to the supply of gas to V^0. The discussion of bubble break-up remains largely unchanged; condition (5.6) becomes

$$\int_{S_0^0}\frac{\partial\bar{\phi}_0}{\partial n}dS = -M, \tag{6.10}$$

and equation (5.8) must be changed correspondingly, but the basic procedure remains the same.

Because gas is being supplied to V^0, bubbles are being produced by necking and breaking of S^0 and the resulting generation of bubble oscillations is described by the same theory as that for division of the bubbles themselves – formally it is the same process.

In practice, we would expect that the cavity which is used to generate bubbles would be much larger than the bubbles themselves. Of course, the above theory is invalid if the cavity size approaches the wavelength for the bubble frequencies in air, but then we would expect there to be many complications in this case. With this restriction in mind, we may now look at the limit of large V^0.

When V_0^0 is large, one of the modes of oscillation, say $\mu = 0$, separates out from the rest in that it has a much lower frequency. We can identify this mode as a cavity resonance of V^0 with the liquid. The frequency of this mode is $\omega_0 = O((V_0^0)^{-\frac{1}{2}})$ for large V_0^0, and Φ_0 satisfies the eigenvalue problem consisting of Laplace's equation and

$$\int_{S_0^0} \frac{\partial \Phi_0}{\partial n} dS = \frac{\omega_0^2 V_0^0}{\gamma} \Phi_0 \quad \text{on} \quad S_0, \tag{6.11}$$

$$\int_{S_0^j} \frac{\partial \Phi_0}{\partial n} dS = 0, \tag{6.12}$$

$$\frac{\partial \Phi_0}{\partial n} = 0 \quad \text{on} \quad S_0, \tag{6.13}$$

and Φ_0 is constant on S_0^j. Thus, the bubbles react incompressibly to the cavity resonance. The normalization condition implies that

$$|\alpha_0^0| \leqslant (V_0^0)^{-\frac{1}{2}} \tag{6.14}$$

a condition which will be used later on.

The remaining modes ($\mu = 1, \ldots, N$) can be identified as bubble resonances and have frequencies $\omega_\mu = O(1)$. The corresponding eigenvalue problem consists of Laplace's equation for Φ_μ and the boundary conditions

$$\Phi_\mu = 0 \quad \text{on} \quad S_0^0, \tag{6.15}$$

$$\int_{S_0^j} \frac{\partial \Phi_\mu}{\partial n} dS = \frac{\omega_\mu^2 V_0^j}{\gamma} \Phi_\mu \quad \text{on} \quad S_0^j, \tag{6.16}$$

$$\frac{\partial \Phi_\mu}{\partial n} = 0 \quad \text{on} \quad S_0. \tag{6.17}$$

Thus, the air-supply cavity reacts to bubble resonances by maintaining its pressure constant.

Turning now to the amplitude equation when V_0^0 is large, for the cavity resonance we see from (6.14) that

$$\alpha_0^j = O[(V_0^0)^{-\frac{1}{2}}], \tag{6.18}$$

and since the total mass supply rate M is to be maintained $O(1)$ we see that (6.9) becomes

$$\frac{da_0}{dt_1} + a_0 \frac{\eta(\gamma-1)\,\omega_0^{\frac{1}{2}}(\alpha_0^0)^2 J_0^0}{2^{\frac{3}{2}}\gamma^{\frac{1}{2}}\epsilon} = 0, \tag{6.19}$$

and so the main cause of damping of this mode is thermal conduction in V^0; the damping takes place over a time scale $O[J_0^0(V_0^0)^{-\frac{4}{3}}]$ in t_1.

In order to obtain a simplification of (6.9) when $\mu \neq 0$ we need to assume that $J_0^0(V_0^0)^{-2}$ is small; this will be the case when V_0^0 is large, unless a very convoluted shape is chosen for V^0. Under this condition we find that

$$\frac{da_\mu}{dt_1} + a_\mu \frac{\omega_\mu^{\frac{1}{2}}\eta(\gamma-1)}{2^{\frac{3}{2}}\gamma^{\frac{1}{2}}\epsilon}\left(\sum_{j=1}^{N}(\alpha_\mu^j)^2 J_0^j\right) = 0, \tag{6.20}$$

and so the presence of the air-supply cavity does not affect the damping of the bubble resonances, this being due to thermal conduction in the bubbles. It may be noted at this point that if one desired to damp bubble oscillations more quickly, then this could be achieved by increasing the surface area J^0 of the cavity so that $J^0(V^0)^{-2}$ is large, for instance by placing baffles inside the cavity.

We now investigate the effects of introducing a free surface into the problem. By a free surface, we mean a gas–liquid interface S^∞ which occupies a finite range in the co-ordinate y and is not closed. This could, for instance, be the surface of a tank of water in which the bubbles are situated. The volume of gas which is in contact with S^∞ will be denoted by V^∞.

The preceding analysis is not greatly changed by this new component of the model. The eigenvalue problem for Φ_μ and ω_μ now has the additional boundary condition

$$\Phi_\mu = 0 \quad \text{on} \quad S_0^\infty, \tag{6.21}$$

while equation (4.2) for the action remains unchanged. There is no thermal boundary layer on S_0^∞ at zeroth order.

The basic flow is governed by the same equations as before, provided we set

$$\bar{F}^\infty = 0 \tag{6.22}$$

in equation (5.3).

This completes our theoretical investigation of bubble oscillations and we now pass on to a few examples of such problems which, although they do not use the full force of the preceding theory, are nevertheless best seen in the light of the above formulation.

7. Examples of bubble oscillations

The calculations involved in examples (*d*), (*f*) and (*g*) are somewhat complicated and the interested reader will find them in appendix B.

(*a*) The first, and simplest, case we consider is that of a spherical gas bubble in an infinite liquid which is at rest at infinity. This problem has often been studied before and we include it here as a simple illustration.

Let the non-dimensionalized radius of the bubble be R; then the eigenvalue problem for the determination of the frequency of oscillation is trivial and leads to

$$\omega^2 = \frac{3\gamma}{R^2} \tag{7.1}$$

(which is the natural frequency of oscillation given by linear theory neglecting surface-tension effects) and

$$\Phi = \frac{1}{(\frac{4}{3}\pi R)^{\frac{1}{2}} r}, \tag{7.2}$$

where r is the distance from the bubble's centre.

The amplitude equation is

$$\frac{da}{dt_1}+\left[\frac{\eta(\gamma-1)}{\epsilon R^{\frac{3}{2}}(3\gamma)^{\frac{1}{4}}}\left(\frac{3}{2}\right)^{\frac{3}{2}}+\frac{3\gamma K}{2\epsilon R}\right]a = 0, \tag{7.3}$$

leading to exponential decay of the action (which in this case is proportional to the amplitude of oscillation because there is no basic flow) under the effects of thermal conduction and radiation. The damping constant implicit in equation (7.3) is just the result of Pfriem for nearly adiabatic oscillations of large gas bubbles (see Devin 1959; Prosperetti 1977).

(*b*) A more interesting case is to consider a spherical bubble to which gas is supplied at constant rate M from a source at its centre when $-T_0 < t_1 < 0$ and to which the supply is cut off at $t_1 = 0$. The basic flow just before $t = 0$ is given by

$$\bar{\phi}_0 = -\frac{M}{4\pi r}, \tag{7.4}$$

while for $t > 0$

$$\bar{\phi}_0 = 0. \tag{7.5}$$

Thus continuity of ϕ_0 about $t = 0$ gives

$$\mathscr{R}\left\{\frac{A(0)}{(\frac{4}{3}\pi R)^{\frac{1}{2}}r}\right\} = -\frac{M}{4\pi r}, \tag{7.6}$$

where we have taken $\Gamma(0) = 0$. Continuity of p_0 yields

$$\mathscr{R}\{iA(0)\} = 0, \tag{7.7}$$

so that

$$A(0) = -M\left(\frac{R}{12\pi}\right)^{\frac{1}{2}}, \tag{7.8}$$

and then the action (or in this case $|A|$ as well) decays according to (7.3).

It will be noted that in this case the entire basic motion for $t < 0$ is converted into oscillations for $t > 0$. Despite the gross approximation of this example it may well model production of a single bubble from a gas supply.

(*c*) Consider next a hemispherical bubble, attached to an infinite plane. The eigenvalue problem for the oscillations is effectively the same as for the spherical bubble – the eigenfrequency is given by (7.1), while

$$\Phi = \frac{1}{(\frac{2}{3}\pi R)^{\frac{1}{2}}r}. \tag{7.9}$$

The amplitude equation is

$$\frac{da}{dt_1}+\left[\frac{3\eta(\gamma-1)}{2\epsilon R^{\frac{3}{2}}(3\gamma)^{\frac{1}{4}}}\left(\frac{3}{2}\right)^{\frac{3}{2}}+\frac{3\gamma K}{4\epsilon R}\right]a = 0, \tag{7.10}$$

so that the decay rate due to thermal damping is $\frac{3}{2}$ times that for the spherical bubble of the same radius, while the decay rate due to radiative damping is one-half of that for the corresponding spherical bubble. The difference in the thermal damping arises because we have adopted a constant-temperature boundary condition on the solid plane, rather than a zero-heat-flux condition there.

(*d*) The normal modes for two spherical bubbles in an infinite liquid can be determined by separation of Laplace's equation in bispherical co-ordinates, to obtain the

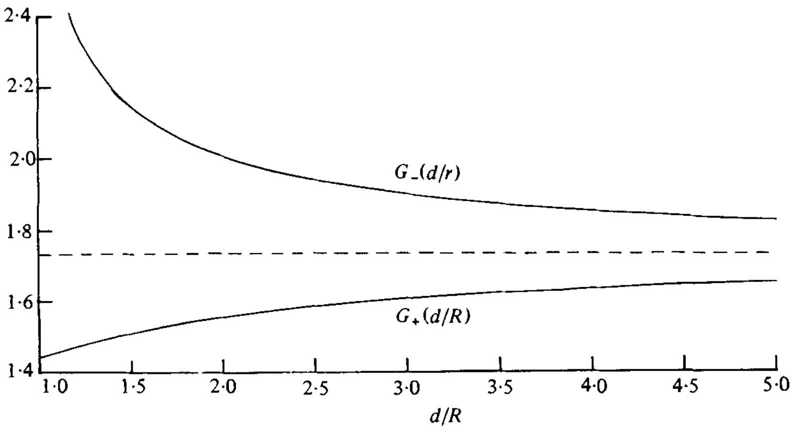

FIGURE 1. Variation of the functions $G_{\pm}(d/R)$.

matrix A_{jk}. In this case the assumed bubble geometry is not an equilibrium state for the basic motion because of the influence of the bubble oscillations on the basic flow. However, if the oscillations are small then the distortion of the bubble shapes from spherical will be negligible. We give the results here for the special case when the two bubbles have the same radius R and there is a distance $2d$ between their centres.

In this case one of the modes of pulsation is symmetric about the plane of symmetry of the bubbles while the other is antisymmetric about the same plane; that is Φ_μ is symmetric (respectively antisymmetric). The frequency of vibration is given by

$$\omega^2 = \frac{6\gamma(d^2/R^2-1)^{\frac{1}{2}}}{R^2}\sum_{n=0}^{\infty}\frac{1}{\Lambda^{2n+1}\pm 1}, \tag{7.11}$$

where

$$\Lambda = \frac{d}{R}+\left(\frac{d^2}{R^2}-1\right)^{\frac{1}{2}}, \tag{7.12}$$

and the choice of signs in (7.11) is dictated by the parity of the mode considered: the + sign for the symmetric mode and the − sign for the antisymmetric one. Thus the antisymmetric mode always has a higher frequency than the symmetric one. The amplitude equation takes the form

$$\frac{da_\mu}{dt_1}+\left[\frac{3\omega_\mu^{\frac{1}{2}}\eta(\gamma-1)}{2^{\frac{3}{2}}\gamma^{\frac{1}{2}}\epsilon R}+\frac{K\omega_\mu^4 R^3(1\pm 1)}{3\gamma\epsilon}\right]a_\mu = 0, \tag{7.13}$$

so that the thermal decay rate for the antisymmetric mode is larger than for the symmetric mode; the radiative decay rate for the antisymmetric mode is zero to this order.

Equation (7.11) can be written in the form

$$\omega = \frac{\gamma^{\frac{1}{2}}}{R}G_{\pm}\left(\frac{d}{R}\right), \tag{7.14}$$

and the functions $G_\pm$ are displayed graphically in figure 1. The function $G_-(d/R)$ has a logarithmic infinity as $d/R \to 1$, while $G_+(1) = (3\ln 2)^{\frac{1}{2}}$.

(*e*) The symmetric solution to the above problem also describes the case of a spherical bubble, radius R, whose centre is a distance d from a rigid plane (cf. the

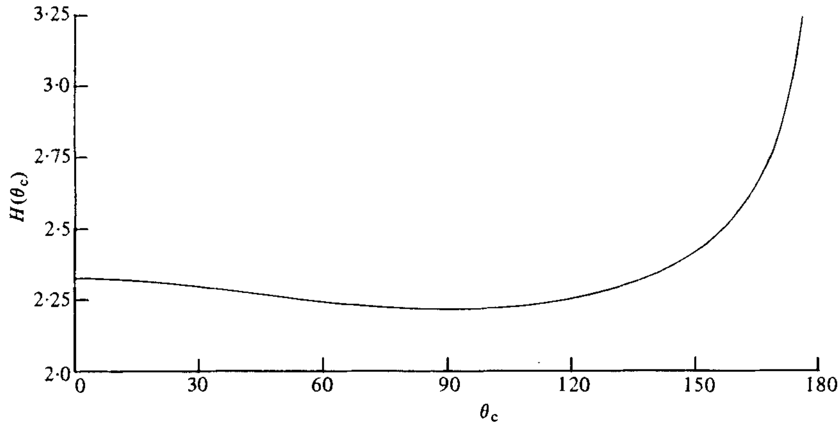

FIGURE 2. Variation of the function $H(\theta_c)$.

method of images in electrostatics) – the frequency and decay rate are found to be identical (the problem for $\hat{\phi}_0$ is easily solved in this case). Similarly, the antisymmetric solution for two spherical bubbles of the same radius describes the case of a spherical bubble, radius R, centred at a distance d from a horizontal, free surface.

A similar procedure to the classification of the modes into symmetric and antisymmetric in the case of two bubbles with the same radius can be carried out whenever the bubble geometry shows symmetry. More precisely, the symmetry group for the geometry induces partitioning of the modes according to representation theory (see Tinkham 1964).

(*f*) Another case of interest is that of an ellipsoidal bubble having semi-axes $b \leqslant c \leqslant d$, for which separation of Laplace's equation in ellipsoidal co-ordinates yields

$$\omega^2 = \frac{3\gamma \sin\chi}{bcF\left(\chi \,\middle|\, \dfrac{d^2-c^2}{d^2-b^2}\right)}, \tag{7.15}$$

where $0 \leqslant \chi \leqslant \frac{1}{2}\pi$ is defined by

$$\cos\chi = \frac{b}{d}, \tag{7.16}$$

and $F(\chi|m)$ is the elliptical integral of the first kind (Abramowitz & Stegun 1964, chap. 17). The equation for the action is determined as before; it involves the area of the ellipsoid, which is also expressible in terms of elliptic integrals.

(*g*) The frequency of oscillation of a spherical bubble of volume V attached to an infinite solid plane at arbitrary contact angle θ_c can be obtained using separation of Laplace's equation in toroidal co-ordinates. Note that the contact angle is here defined so that $\theta_c = 0$ corresponds to the bubble just touching the plane, while $\theta_c = \frac{1}{2}\pi$ corresponds to a hemispherical bubble. We find that

$$\omega = \frac{\gamma^{\frac{1}{2}}}{V^{\frac{1}{2}}} H(\theta_c) \tag{7.17}$$

with $H(\theta_c)$ as shown in figure 2. The function H has an algebraic infinity at $\theta_c = \pi$, but varies remarkably little over most of the range between $\theta_c = 0$ and $\theta_c = \pi$. The amplitude equation is now easily obtained.

(*h*) Next consider excitation of a spherical bubble of radius R by a source of frequency ω at a distance d away from its centre. The problem satisfied by $\tilde{\phi}_0$ consists of equation (3.16) with

$$\nabla^2\tilde{\phi}_0 = 0, \tag{7.18}$$

$$\tilde{\phi}_0 \sim \mathscr{R}\left\{\frac{Be^{-i\omega t}}{D}\right\} \quad \text{as} \quad D \to 0, \tag{7.19}$$

where D is the distance from the source and B measures the source strength.

We set

$$\tilde{\phi}_0 = \mathscr{R}\{\Phi(\mathbf{x})\, e^{-i\omega t}\} \tag{7.20}$$

to obtain

$$\nabla^2\Phi = 0, \tag{7.21}$$

$$\Phi \sim \frac{B}{D} \quad \text{as} \quad D \to 0, \tag{7.22}$$

$$\frac{3\gamma}{4\pi R^3}\int_{S_0}\frac{\partial\Phi}{\partial n}dS = \omega^2\Phi \quad \text{on} \quad S_0. \tag{7.23}$$

This problem has the electrostatic analogue of a point charge and a conducting sphere, which is soluble by the method of images; we find that

$$\Phi = \frac{B}{d(1-\omega^2/\omega_r^2)} \quad \text{on} \quad S_0, \tag{7.24}$$

where ω_r is the resonance frequency of the bubble as given by equation (7.1). The behaviour of Φ at large distances is given by

$$\Phi \sim \frac{B}{D}\left[1+\frac{R}{d}\left(\frac{\omega^2}{\omega_r^2-\omega^2}\right)\right], \tag{7.25}$$

corresponding to an effective source of strength

$$B\left[1+\frac{R}{d}\left(\frac{\omega^2}{\omega_r^2-\omega^2}\right)\right].$$

Thus the bubble increases the effect of the source for frequencies less than $\omega_r(1-R/2d)^{-\frac{1}{2}}$ and decreases its effect at large distances for frequencies higher than this. The effective source strength is zero for $\omega = \omega_r(1-R/d)^{-\frac{1}{2}}$.

The singularity which occurs in the solution at resonance will be removed by damping, of course; formally we would suppose that $\omega-\omega_r = O(\epsilon)$, $B = O(\epsilon)$ (to maintain the response of $O(1)$) and then derive an amplitude equation for $A(t_1)$ in

$$\tilde{\phi}_0 = \mathscr{R}\{A(t_1)\,\Phi(\mathbf{x})\, e^{-i\omega_r t}\}, \tag{7.26}$$

as in §§3 and 4 (Nayfeh & Mook 1979). The procedure is complicated by the $O(\epsilon)$ shift in resonance frequency, corresponding to variation of τ (defined in (4.26)), which we have already noted. We do not give the details here.

Returning to the problem of a spherical bubble and a rigid plane, it may seem somewhat paradoxical that, as $d/R \to \infty$, the radiative decay rate approaches twice the value which it would have if the bubble were in an infinite liquid. This results because we have implicitly assumed that d is of $O(1)$; the results for d of the order of the wavelength will now be derived.

In this case, ω and Φ are given by (7.1) and (7.2), the rigid plane now being in the outer (radiation) region. If $\hat{r}$ is the distance, in terms of $\hat{\mathbf{x}}$, from the bubble, while $\hat{r}_{\mathrm{im}}$ is the distance from the reflected image of the bubble in the plane, then

$$\hat{\Phi} = \frac{A}{(\frac{4}{3}\pi R)^{\frac{1}{2}}}\left(\frac{e^{i\omega\hat{r}}}{\hat{r}} + \frac{e^{i\omega\hat{r}_{\mathrm{im}}}}{\hat{r}_{\mathrm{im}}}\right), \tag{7.27}$$

so that matching yields the relation

$$\Phi^{(1)} \rightarrow \frac{i\omega K A}{\epsilon(\frac{4}{3}\pi R)^{\frac{1}{2}}}\left(1 + \frac{e^{i\omega K d}}{i\omega K d}\right), \tag{7.28}$$

as $r \rightarrow \infty$. Following the same arguments as before, we arrive at the equation for the action

$$\frac{da}{dt_1} + \left[\frac{\eta(\gamma-1)}{\epsilon R^{\frac{3}{2}}(3\gamma)^{\frac{1}{2}}}\left(\frac{3}{2}\right)^{\frac{3}{2}} + \frac{3\gamma K}{2\epsilon R}\left(1 + \frac{\sin(\omega K d)}{\omega K d}\right)\right] a = 0, \tag{7.29}$$

where it is seen that the radiative decay rate is that for a bubble without a rigid surface, multiplied by the factor $1 + \sin(\omega K d)/\omega K d$, which starts off at the value 2 when d is small compared to the wavelength and oscillates about the value 1, which is its limit as $Kd \rightarrow \infty$. Thus, as expected, the radiative decay rate approaches its value without a plate, but it does so much more slowly than, say, the frequency of oscillation of the bubble.

The case of a spherical bubble and a horizontal free surface is similar; the factor $1 + \sin(\omega K d)/\omega K d$ is changed to $1 - \sin(\omega K d)/\omega K d$, so that the radiative decay rate is zero when d is small compared to the wavelength and tends to its value without the free surface in an oscillatory manner as $Kd \rightarrow \infty$.

Similar results for the radiative decay rate are expected whenever the bubbles are within a few wavelengths of some non-compact object.

8. Conclusion

We have provided a method for determining the normal frequencies and modes of a collection of gas bubbles in a liquid in the presence of buoyancy, surface tension and solid boundaries. By passing to the next order we have also found the equation governing the decay of action for each mode.

The basic flow, which determines the geometry in which the bubble pulsations occur, is described by a certain incompressible velocity potential $\overline{\phi}_0$ which satisfies boundary conditions on the zeroth-order bubble surfaces S_0^j – these surfaces being convected by the basic flow. In general, the presence of the oscillations affects the basic motion, and analytical solutions for the basic flow are not possible. Numerical solution of the basic-flow equations is, however, quite feasible and more efficient than solution of the complete equations because much computing time would be spent following the detailed, rapid bubble oscillations, whereas the present approach effectively averages over these rapid variations.

It is possible to introduce a variety of other components into the model, e.g. elastic plates could result in damping of the bubble pulsations through the generation of waves in the plate.

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Appendix A

Going to the next order we find that

$$\nabla p_1^j = -\frac{\delta}{\epsilon}\frac{\partial \mathbf{u}_0^j}{\partial t_0}, \tag{A 1}$$

and so

$$p_1^j = -\frac{\delta}{\epsilon}\frac{\partial \phi_0^j}{\partial t_0} + F^j(t), \tag{A 2}$$

$$\bar{p}_1^j = \bar{F}^j(t_1), \tag{A 3}$$

$$\tilde{p}_1^j = -\frac{\delta}{\epsilon}\frac{\partial \tilde{\phi}_0^j}{\partial t_0} + \tilde{F}^j(t). \tag{A 4}$$

We will determine $\tilde{F}^j$ by using the integral relation (2.21).

To do this, note that

$$\int_{V^j} p^j\,dv = (V_0^j + \epsilon V_1^j)p_0^j + \epsilon F^j V_0^j + o(\epsilon), \tag{A 5}$$

which we differentiate with respect to t, giving

$$\frac{d}{dt}\int_{V^j} p^j\,dv = V_0^j\frac{dp_0^j}{dt_0} + \epsilon\left(\frac{d}{dt_0}(p_0^j V_1^j) + \frac{d\tilde{F}^j}{dt_0}V_0^j + \mathscr{R}\left\{i\sum_{\mu=1}^{N}\frac{d}{dt_1}(\omega_\mu A_\mu \alpha_\mu^j)\exp(-i\Gamma_\mu/\epsilon)\right\}V_0^j\right) + o(\epsilon), \tag{A 6}$$

so that (2.21) becomes

$$V_0^j\frac{dp_0^j}{dt_0} + \epsilon\left(\frac{dp_0^j}{dt_0}V_1^j + \frac{d\tilde{F}^j}{dt_0} + \mathscr{R}\left\{i\sum_{\mu=1}^{N}\frac{d}{dt_1}(\omega_\mu A_\mu \alpha_\mu^j)\exp(-i\Gamma_\mu/\epsilon)\right\}V_0^j\right) - \gamma(1+\epsilon p_0^j)\int_{S^j}\frac{\partial \phi^j}{\partial n}dS = -\eta^2\int_{J^j}\frac{\partial \theta^j}{\partial n}dS + o(\epsilon). \tag{A 7}$$

Now we use

$$\int_{S^j}\frac{\partial \phi^j}{\partial n}dS = \int_{S^j}\frac{\partial \phi}{\partial n}dS = \int_{S_0^j}\frac{\partial \phi}{\partial n}dS, \tag{A 8}$$

obtained from (2.6) and (2.16), to estimate

$$\int_{S^j}\frac{\partial \phi^j}{\partial n}dS = \int_{S_0^j}\left(\frac{\partial \phi_0}{\partial n} + \epsilon\frac{\partial \phi_1}{\partial n}\right)dS + o(\epsilon). \tag{A 9}$$

The right-hand side of (A 7) can in turn be written

$$-\eta^2\int_{J^j}\frac{\partial \theta^j}{\partial n}dS = -\frac{(\gamma-1)}{\gamma}\eta J_0^j \mathscr{R}\left\{\sum_\mu i\omega_\mu \sigma_\mu A_\mu \alpha_\mu^j \exp(-i\Gamma_\mu/\epsilon)\right\} + o(\epsilon), \tag{A 10}$$

so that (A 7), (A 9) and (A 10) give

$$\frac{d\tilde{F}^j}{dt_0}V_0^j = \gamma\int_{S_0^j}\frac{\partial \phi_1}{\partial n}dS - \bar{V}_1^j\frac{dp_0^j}{dt_0} + V_0^j\frac{(\gamma+1)}{\gamma}p_0^j\frac{dp_0^j}{dt_0} - V_0^j\mathscr{R}\left\{\sum_{\mu=1}^{N} i\frac{d}{dt_1}(\omega_\mu A_\mu \alpha_\mu^j)\exp(-i\Gamma_\mu/\epsilon)\right\} - \frac{(\gamma-1)\eta}{\gamma\epsilon}J_0^j\mathscr{R}\left\{\sum_\mu i\omega_\mu\sigma_\mu A_\mu \alpha_\mu^j\exp(-i\Gamma_\mu/\epsilon)\right\}, \tag{A 11}$$

which is the required equation for $\tilde{F}^j$.

At the next order, we have in the liquid

$$\nabla^2\phi_1 = 0, \tag{A 12}$$

$$p_1 = -\tfrac{1}{2}|\nabla\phi_0|^2 - \frac{\partial\phi_1}{\partial t_0} - \frac{\partial\overline{\phi}_0}{\partial t_1} - \mathscr{R}\left\{\sum_{\mu=1}^{N}\frac{\partial}{\partial t_1}(A_\mu\Phi_\mu)\exp(-i\Gamma_\mu/\epsilon)\right\} - \frac{\Delta y}{\epsilon}, \tag{A 13}$$

so that

$$\tilde{p}_1 = -\nabla\overline{\phi}_0 . \nabla\tilde{\phi}_0 - \tfrac{1}{2}(|\nabla\tilde{\phi}_0|^2 - \overline{|\nabla\tilde{\phi}_0|^2}) - \frac{\partial\tilde{\phi}_1}{\partial t_0} - \mathscr{R}\left\{\sum_{\mu=1}^{N}\frac{\partial}{\partial t_1}(A_\mu\Phi_\mu)\exp(-i\Gamma_\mu/\epsilon)\right\}. \tag{A 14}$$

The condition (2.19) becomes

$$p_1^j = p_1 - \frac{\overline{f_1^j\nabla f_0^j . \nabla p_0}}{|\nabla f_0^j|^2} + \left(\frac{1}{R_0^j} + \frac{1}{R_0'^j}\right) \quad \text{on} \quad S_0^j, \tag{A 15}$$

so that

$$\tilde{p}_1^j = \tilde{p}_1 - \frac{\nabla f_0^j . (\tilde{f}_1^j\nabla\tilde{p}_0 - \overline{\tilde{f}_1^j\nabla\tilde{p}_0} + \overline{f}_1^j\nabla\tilde{p}_0)}{|\nabla f_0^j|^2} \quad \text{on} \quad S_0^j. \tag{A 16}$$

At order ϵ, equation (2.15) yields the results

$$\frac{\partial f_0^j}{\partial t_1} + \nabla\overline{\phi}_0 . \nabla f_0^j = 0, \quad \frac{\partial \tilde{f}_1^j}{\partial t_0} + \nabla\tilde{\phi}_0 . \nabla f_0^j = 0. \tag{A 17), (A 18}$$

Next, combining (A 4), (A 11), (A 14), (A 16), (A 17) and (A 18) we obtain

$$\begin{aligned}\frac{\partial^2\tilde{\phi}_1}{\partial t_0^2} + \frac{\gamma}{V_0^j}\int_{S_0^j}\frac{\partial\tilde{\phi}_1}{\partial n}dS = \mathscr{R}\Bigg\{\sum_{\mu=1}^{N}\Bigg[&i\left(\omega_\mu\frac{d}{dt_1}(A_\mu\alpha_\mu^j)\right.\\ &\left.+\frac{d}{dt_1}(A_\mu\omega_\mu\alpha_\mu^j)\right) + \frac{(\gamma-1)\eta}{\gamma\epsilon}\frac{J_0^j}{V_0^j}i\omega_\mu\sigma_\mu A_\mu\alpha_\mu^j - \frac{\delta\omega_\mu^2}{\epsilon}\Phi_\mu^j A_\mu\\ &+\frac{\overline{V}_1^j}{V_0^j}\omega_\mu^2 A_\mu\alpha_\mu^j - \overline{f}_1^j\omega_\mu^2 A_\mu\frac{\nabla f_0^j . \nabla\Phi_\mu}{|\nabla f_0^j|^2}\Bigg]\exp(-i\Gamma_\mu/\epsilon)\\ &+\tfrac{1}{2}i\sum_{\mu,\nu=1}^{N}(A_\mu A_\nu(\omega_\mu+\omega_\nu)\Big[\tfrac{1}{2}\nabla\Phi_\mu . \nabla\Phi_\nu\\ &+\frac{\omega_\mu}{\omega_\nu}\frac{\partial\Phi_\mu}{\partial n}\frac{\partial\Phi_\nu}{\partial n} - \frac{(\gamma-1)}{2\gamma}\omega_\mu\omega_\nu\alpha_\mu^j\alpha_\nu^j\Big]\exp(-i\Gamma_\mu+\Gamma_\nu)/\epsilon)\\ &+A_\mu A_\nu^*(\omega_\mu-\omega_\nu)\Big[\tfrac{1}{2}\nabla\Omega_\mu . \nabla\Phi_\nu - \frac{\omega_\mu}{\omega_\nu}\frac{\partial\Phi_\mu}{\partial n}\frac{\partial\Phi_\nu}{\partial n}\\ &+\frac{(\gamma+1)}{2\gamma}\omega_\mu\omega_\nu\alpha_\mu^j\alpha_\nu^j\Big]\exp(-i(\Gamma_\mu-\Gamma_\nu)/\epsilon)\Bigg\}.\end{aligned} \tag{A 19}$$

Here, the double sum corresponds to the production of sum and difference frequencies by nonlinear effects. To avoid secular terms in $\tilde{\phi}_1$, we assume a solution of the form

$$\begin{aligned}\tilde{\phi}_1 = \mathscr{R}\Bigg\{\sum_{\mu=1}^{N}\Phi_\mu^{(1)}\exp(-i\Gamma_\mu/\epsilon) + \sum_{\mu,\nu=1}^{N}(\Phi_{\mu\nu}^{(+)}\exp(-i(\Gamma_\mu+\Gamma_\nu)/\epsilon)&\\ +\Phi_{\mu\nu}^{(-)}\exp(-i(\Gamma_\mu-\Gamma_\nu)/\epsilon))\Bigg\}.&\end{aligned} \tag{A 20}$$

Substituting this form into equation (A 19) and assuming that conditions (i) and (ii) stated in §4 are satisfied we find that

$$-\omega_\mu^2\Phi_\mu^{(1)}+\frac{\gamma}{V_0^j}\int_{S_0^j}\frac{\partial\Phi_\mu^{(1)}}{\partial n}dS = i\left(\omega_\mu\frac{d}{dt_1}(A_\mu\alpha_\mu^j)+\frac{d}{dt_1}(A_\mu\omega_\mu\alpha_\mu^j)\right)$$
$$+\frac{(\gamma-1)J_0^j}{\gamma\epsilon\;V_0^j}i\omega_\mu\sigma_\mu A_\mu\alpha_\mu^j-\frac{\delta\omega_\mu^2}{\epsilon}\Phi_\mu^j A_\mu$$
$$+\frac{\overline{V}_1^j}{V_0^j}\omega_\mu^2 A_\mu\alpha_\mu^j-\bar{f}_1^j\omega_\mu^2 A_\mu\frac{\nabla f_0^j.\nabla\Phi_\mu}{|\nabla f_0^j|^2}\quad\text{on}\quad S_0^j. \tag{A 21}$$

while

$$\nabla^2\Phi_\mu^{(1)} = 0 \quad\text{in}\quad V_0, \tag{A 22}$$

$$\frac{\partial\Phi_\mu^{(1)}}{\partial n} = 0 \quad\text{on}\quad S_0. \tag{A 23}$$

Solubility of the problem consisting of (A 21)–(A 23) requires that the right-hand side of (A 21) satisfy an orthogonality condition which, as usual in such problems, provides the amplitude equation. If we denote the right-hand side of (A 21) by F_μ^j then this condition is

$$\sum_{j=1}^{N}\int_{S_0^j}F_\mu^j\frac{\partial\Phi_\mu}{\partial n}dS = 0. \tag{A 24}$$

Substituting the expression for F_μ^j into (A 24), using (3.22) and the normalization condition for β_μ^j we obtain the amplitude equation

$$2i\omega_\mu\frac{dA_\mu}{dt_1}+iA_\mu\frac{d\omega_\mu}{dt_1}+\frac{i(\gamma-1)\,\omega_\mu\sigma_\mu\eta A_\mu}{\gamma\epsilon}\sum_{j=1}^{N}(\alpha_\mu^j)^2J_0^j$$
$$+\omega_\mu^2A_\mu\sum_{j=1}^{N}(\alpha_\mu^j)^2\overline{V}_1^j-\gamma A_\mu\sum_{j=1}^{N}\int_{S_0^j}\left(\frac{\delta}{\epsilon}\Phi_\mu^j+\bar{f}_1^j\frac{\nabla f_0^j.\nabla\Phi_\mu}{|\nabla f_0^j|^2}\right)\frac{\partial\Phi_\mu}{\partial n}dS = 0, \tag{A 25}$$

which describes the development of A_μ with the slow time scale t_1.

Appendix B

In this appendix we give brief outlines of the calculations involved in examples (*d*), (*f*) and (*g*) of §7. The symbols used for all variables, apart from γ, in this appendix will be independent of those used in the main text unless otherwise stated.

(*d*) For the case of two spherical bubbles, both of radius R at a distance $2d$ apart we use bispherical co-ordinates μ, η, ϕ as given on p. 1298, vol. 2 of Morse & Feshbach (1953). The bubbles are situated on $\mu = \mu_0$, $-\mu_0$, where $\cosh\mu_0 = d/R$, $\mu_0 > 0$.

Two solutions of Laplace's equation are given by

$$\Phi_\pm = 2^{\frac{1}{2}}(\cosh\mu-\cos\eta)^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{e^{(n+\frac{1}{2})\mu}\pm e^{-(n+\frac{1}{2})\mu}}{e^{(2n+1)\mu_0}\pm 1}P_n(\cos\eta). \tag{B 1}$$

Both satisfy $\Phi_\pm = 1$ on $\mu = \mu_0$, while Φ_+ is symmetric and Φ_- is anti-symmetric about $\mu = 0$. It can then be shown that

$$\int\frac{\partial\Phi_\pm}{\partial n}dS = 8\pi R\sinh\mu_0\sum_{n=0}^{\infty}\frac{1}{e^{(2n+1)\mu_0}\pm 1}, \tag{B 2}$$

where the surface integral is taken over the sphere $\mu = \mu_0$ and the normal is taken into the bubble. From equation (3.22) we have

$$\omega^2 = \frac{3\gamma}{4\pi R^3}\int \frac{\partial \Phi_\pm}{\partial n}\, dS, \tag{B 3}$$

giving the frequency ω of the normal modes. Combining (B 2) and (B 3) we obtain

$$\omega^2 = \frac{6\gamma}{R^2}\sinh \mu_0 \sum_{n=0}^{\infty} \frac{1}{e^{(2n+1)\mu_0} \pm 1}, \tag{B 4}$$

which gives equation (7.11) via the definition of μ_0.

(*f*) For the problem of the ellipsoidal bubble, having semi-axes $\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3$, we define $a = (\alpha_3^2-\alpha_1^2)^{\frac{1}{2}}$, $b = (\alpha_3^2-\alpha_2^2)^{\frac{1}{2}}$, $c = \alpha_3$. It is shown on p. 1308, vol. 2 of Morse & Feshbach (1953) that, in the electrostatic analogue of the oscillation problem, the capacitance of the ellipsoid is given by

$$C = \frac{a}{F\left(\chi \middle| \frac{b^2}{a^2}\right)}, \tag{B 5}$$

where χ and $F(\chi|m)$ are as in §7. The frequency of the bubble is therefore given by

$$\omega^2 = \frac{3\gamma C}{\alpha_1\alpha_2\alpha_3} = \frac{3\gamma \sin \chi}{\alpha_1\alpha_2 F\left(\chi \middle| \frac{\alpha_3^2-\alpha_2^2}{\alpha_3^2-\alpha_1^2}\right)}, \tag{B 6}$$

which corresponds to equation (7.15).

(*g*) For a spherical bubble of volume V, attached to a solid plane at contact angle θ_c, we use toroidal co-ordinates, μ, η, ϕ, as defined on p. 1301, vol. 2 of Morse & Feshbach (1953). The plane corresponds to $\eta = 0$, while the bubble is given by $\eta = \theta_c$. A solution of Laplace's equation is

$$\Phi = 1 - \frac{2^{\frac{3}{2}}}{\theta_c}(\cosh \mu - \cos \eta)^{\frac{1}{2}} \sum_{n=0}^{\infty} Q_{\lambda(n+\frac{1}{2})-\frac{1}{2}}(\cosh \mu) \cos \lambda\,(n+\tfrac{1}{2})\,\eta, \tag{B 7}$$

where $\lambda = \pi/\theta_c$; $\Phi = 1$ on $\eta = \theta_c$, $\partial\Phi/\partial\eta = 0$ on $\eta = 0$ and $\Phi \to 0$ as r, the distance from the bubble, becomes large (i.e. μ, $\eta \to 0$).

For large r, we expect the solution Φ to have the form

$$\Phi \sim \frac{q}{r}, \tag{B 8}$$

and then we know that

$$\int \frac{\partial \Phi}{\partial n}\, dS = 2\pi q, \tag{B 9}$$

where the integral is taken over the bubble surface and the normal is directed into the bubble. The frequency of oscillation is therefore given by

$$\omega^2 = \frac{2\pi q \gamma}{V}. \tag{B 10}$$

The problem is now to calculate q.

Using formula (5.3.29) of Morse & Feshbach (1953) we obtain the following integral representation:

$$\Phi = 1 - \frac{(\xi-1)^{\frac{1}{2}}}{2^{\frac{1}{2}(\lambda-2)}\theta_c}\int_{-1}^{1}\frac{(1-t^2)^{\frac{1}{2}(\lambda-1)}\,dt}{(\xi-t)^{\frac{1}{2}(\lambda+1)}\left[1-\left(\dfrac{1-t^2}{2(\xi-t)}\right)^{\lambda}\right]}, \tag{B 11}$$

where $\eta = 0$ and $\xi = \cosh\mu$. As $r \to \infty$, i.e. $\mu \to 0$, the integral becomes singular, but by using asymptotic matching methods we find that

$$\Phi \sim \frac{2a}{\theta_c r}\left(\frac{1}{\lambda} - 2\int_0^1\left[\frac{(1-x)^{\frac{1}{2}(\lambda-1)}}{x(1-(1-x)^{\lambda})} - \frac{1}{\lambda x^2}\right]dx\right), \tag{B 12}$$

where

$$a = \sin\theta_c\left[\frac{3V}{\pi(3\cos\theta_c - \cos^3\theta_c + 2)}\right]^{\frac{1}{3}} \tag{B 13}$$

is the radius of the circle of contact of the bubble with the plane. The derivation of equation (B 12) requires the use of three regions in $x = \frac{1}{2}(1-t)$, given by $x = O(1)$, $x = O((\xi-1)^{\frac{1}{2}})$ and $x = O(\xi - 1)$. Finally, using (B 10) and (B 12) we have

$$\omega^2 = \frac{4\pi\gamma a}{\theta_c V}\left(\frac{1}{\lambda} - 2\int_0^1\left[\frac{(1-x)^{\frac{1}{2}(\lambda-1)}}{x(1-(1-x)^{\lambda})} - \frac{1}{\lambda x^2}\right]dx\right). \tag{B 14}$$

Figure 2 was obtained by numerical evaluation of the integral appearing in (B 14); some care is needed in this calculation near the endpoints $x = 0, 1$ and when λ is large.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Washington D.C.: National Bureau of Standards.

CRUM, L. A. 1975 Bjerknes forces on bubbles in a stationary sound field. *J. Acoust. Soc. Am.* **57**, 1363–70.

DEVIN, C. 1959 Survey of thermal, radiation and viscous damping of pulsating air bubbles in water. *J. Acoust. Soc. Am.* **31**, 1654–1667.

FITZPATRICK, H. M. & STRASBERG, M. 1957 Hydrodynamic sources of sound. *Naval Hydrodynamics Pub.* 515, pp. 241–275. National Academy of Sciences, U.S. National Research Council.

FOLDY, L. L. 1945 The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers. *Phys. Rev.* **67**, 107–119.

GRIMSHAW, R. & ALLEN, J. S. 1979 Linearly coupled, slowly varying oscillators. *Stud. Appl. Math.* **61**, 55–71.

MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*. McGraw-Hill.

NAYFEH, A. H. 1973 *Perturbation Methods*. Wiley.

NAYFEH, A. H. & MOOK, D. T. 1979 *Nonlinear oscillations*. Wiley.

PLESSET, M. S. & PROSPERETTI, A. 1977 Bubble dynamics and cavitation. *Ann. Rev. Fluid Mech.* **9**, 145–185.

PROSPERETTI, A. 1977 Thermal effects and damping mechanisms in forced radial oscillations of gas bubbles in liquids. *J. Acoust. Soc. Am.* **61**, 17–27.

TINKHAM, M. 1964 *Group Theory and Quantum Mechanics*. McGraw-Hill.

VAN DYKE, M. D. 1975 *Perturbation Methods in Fluid Mechanics*. Parabolic.

WHITFIELD, O. J. & HOWE, M. S. 1976 The generation of sound by two-phase nozzle flows and its relevance to excess noise of jet engines. *J. Fluid Mech.* **75**, 553–576.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.

VAN WIJNGAARDEN, L. 1972 One-dimensional flow of liquids containing small gas bubbles. *Ann. Rev. Fluid Mech.* **4**, 369–396.